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On the Diophantine Equation $5^x + p^m n^y = z^2$

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ABSTRACT

Diophantine equation is a polynomial equation with two or more unknowns for which only integral solutions are sought. This paper concentrates on finding the integral solutions to the Diophantine equation $5^x + p^m n^y = z^2$ where $p > 5$ a prime number and $y = 1, 2$. The positive integral solutions to the equation are $(x, m, n, y, z) = (2r, t, p^t k^2 \pm 2k5^r, 1, p^t k \pm 5^r)$ and $\left(2r, 2t, \dfrac{5^{2r-\alpha} - 5^{\alpha}}{2p^t}, 2, \dfrac{5^{2r-\alpha} + 5^{\alpha}}{2}\right)$ for $k, r, t \in \mathbb{N}$ where $0 \leq \alpha < r$.

1. Introduction

Diophantine equation has been studied by many authors with different type of equations. Sroysang (2012) showed that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1, 0, 2)$.

Meanwhile, Liu (2013) proved that if $n > 3$ and $p \equiv 3 (\text{mod } 4)$, then the equation $x^4 - q^4 = py^n$ has no positive integer solution (x, y) satisfying gcd $(x, y) = 1$ and $2 \nmid y$ for n, p and q be odd primes. Chotchaisthit (2013) studied on the Diophantine equation $p^x + (p+1)^y = z^2$ where p is a Mersenne prime and found that $(p, x, y, z) = (7, 0, 1, 3), (3, 2, 2, 5)$ are the only solutions for the equation.

Sroysang (2013b) and Sroysang (2013a) proved that the Diophantine equations $5^x + 7^y = z^2$ and $5^x + 23^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integer.

Tatong and Suvarnamani (2015) found that the Diophantine equation $(p+1)^{2x} + q^y = z^2$ has no non-negative integer solution where p is a Mersenne prime number which $q - p = 2$ and x, y, z are non-negative integers. In the same year, Bacani and Rabago (2015) showed that the Diophantine equation $p^x + q^y = z^2$ has infinitely many solutions in positive integer (p, q, x, y, z) where p and q are twin primes. They also found that if the sum of p and q is a square, then the equation has unique solution $(x, y, z) = (1, 1, \sqrt{p+q})$.

The Diophantine equation of the form $p^a + (p+1)^b = z^2$ also studied by Trojovský (2015) and proved that if $p > 3$ then the Diophantine equation $p^a + (p+1)^b = z^2$ does not have integer solution with $b \geq 2$ and z even, and also proved for Diophantine equaton $p^a + (p+1)^b = z^4$. If $p > 2$ then the Diophantine equation does not have integer solution for $b \geq 7$.

This paper concentrates on finding the integral solutions to the Diophantine equation $5^x + p^m n^y = z^2$ where $p > 5$ a prime number and $y = 1, 2$. In order to solve the equation, we will consider the following definition and theorem that can be found in Mollin (2008) and Nagell (1964):

Definition 1.1. : *If u and v are integers, we say that u divides v (denoted as $u|v$) if there exists an integer w such that $v = uw$. If no such w exists, then u does not divide v (denoted by $u \nmid v$). If u divides v, we say that u is a divisor of v, and v is divisible by u.*

Theorem 1.1. : *The bound for the fundamental solution* (u, v) *for the equation* $u^2 - Dv^2 = N$ *is*

$$0 \leq v \leq \frac{y_1}{\sqrt{2(x_1+1)}}\sqrt{N},$$

$$0 <\mid u \mid\leq \sqrt{\frac{1}{2}(x_1+1)N},$$

where N *is positive integer with* (x_1, y_1) *is the fundamental solution of equation* $x^2 - Dy^2 = 1$ *and* D *is natural number which is not a perfect square.*

2. Results and Discussion

In this section, we will discuss on finding the integral solutions to the Diophantine equation $5^x + p^m n^y = z^2$. Firstly, we let $y = 1$ follow by $y = 2$ as in Theorems 2.1 and 2.2 respectively.

Theorem 2.1. : *Let* x, m, n, y, z *be positive integers and* $p > 5$ *a prime number. If* x *is an even number and* $y = 1$*, then the Diophantine equation* $5^x + p^m n^y = z^2$ *has positive integral solutions in form of:*

$$(x, m, n, y, z) = (2r, t, p^t k^2 \pm 2k5^r, 1, p^t k \pm 5^r)$$

where $r, t, k \in \mathbb{N}$.

Proof. Given the Diophantine equation $5^x + p^m n^y = z^2$. We let $y = 1$. Suppose x is an even number, such that $x = 2r$ where $r \in \mathbb{N}$, we have

$$5^{2r} + p^m n = z^2. \tag{1}$$

From (1), we have

$$(z + 5^r)(z - 5^r) = p^{m-\beta} p^\beta n \tag{2}$$

where $0 \leq \beta \leq m$.
Since the LHS must be equal to RHS, we will consider all possible combinations of (2), as follows:
From (i) and (iii), we have

$$\begin{aligned} z \pm 5^r &= p^{m-\beta} n \\ z \mp 5^r &= p^\beta. \end{aligned} \tag{3}$$

By solving the above equations simultaneously, we obtain

$$z = \frac{p^{m-\beta} n + p^\beta}{2}. \tag{4}$$

Table 1: Possible combinations of (2).

i	$z + 5^r = p^{m-\beta}n,$	$z - 5^r = p^{\beta}$
ii	$z + 5^r = p^{m-\beta},$	$z - 5^r = p^{\beta}n$
iii	$z + 5^r = p^{\beta},$	$z - 5^r = p^{m-\beta}n$
iv	$z + 5^r = p^{\beta}n,$	$z - 5^r = p^{m-\beta}$
v	$z + 5^r = p^{m},$	$z - 5^r = n$
vi	$z + 5^r = n,$	$z - 5^r = p^{m}$
vii	$z + 5^r = p^{m}n,$	$z - 5^r = 1$
viii	$z + 5^r = 1,$	$z - 5^r = p^{m}n$

Substitute (4) into (3), we obtain

$$n = p^{2\beta-m} \pm 2p^{\beta-m}5^r. \tag{5}$$

where $\beta \geq m$. This is contradicts since $0 \leq \beta \leq m$.

From (ii) and (iv), we have

$$\begin{aligned} z \pm 5^r &= p^{m-\beta} \\ z \mp 5^r &= p^{\beta}n. \end{aligned} \tag{6}$$

By solving the above equations simultaneously, we obtain

$$z = \frac{p^{m-\beta} + p^{\beta}n}{2}. \tag{7}$$

Substitute (7) into (6), we obtain

$$n = \frac{p^{m-\beta} \pm 2(5^r)}{p^{\beta}}. \tag{8}$$

Substitute (8) into (7), we obtain

$$z = p^{m-\beta} \pm 5^r \tag{9}$$

where $m > \beta$.

From (v) and (vi), we have

$$\begin{aligned} z \pm 5^r &= p^{m} \\ z \mp 5^r &= n. \end{aligned} \tag{10}$$

From (10), we have

$$z = p^m \mp 5^r. \tag{11}$$

By the equations (9) and (11), we obtain

$$z = p^m \pm 5^r \tag{12}$$

where $\beta = 0$ and $m > 0$ since $m > \beta$.

From (vii) and (viii), we have

$$\begin{aligned} z \pm 5^r &= p^m n \\ z \mp 5^r &= 1. \end{aligned} \tag{13}$$

From (13), we have

$$z = 1 \pm 5^r. \tag{14}$$

From (12) and (14), we have

$$z = p^m \pm 5^r$$

where $m \geq 0$. This is contradicts since $m > 0$.

Therefore, from (12), clearly that

$$p^m \mid z \mp 5^r$$

By applying the concept of divisibility (Definition 1.1), there exist k such that $z \mp 5^r = p^m k$ where $k \in \mathbb{N}$. Therefore

$$z = p^m k \pm 5^r$$

Let $m = t \in \mathbb{N}$, we obtain

$$z = p^t k \pm 5^r \tag{15}$$

Substitute (15) into (1), we obtain

$$n = p^t k^2 \pm 2k5^r$$

where $k, r, t \in \mathbb{N}$. □

Theorem 2.2. : *Let x, m, n, y, z be positive integers and $p > 5$ a prime number. If x is an even number and $y = 2$, then the positive integral solutions to the Diophantine equation $5^x + p^m n^y = z^2$ are in the form of*

$$(x, m, n, y, z) = \left(2r, 2t, \frac{5^{2r-\alpha} - 5^\alpha}{2p^t}, 2, \frac{5^{2r-\alpha} + 5^\alpha}{2}\right)$$

where $0 \leq \alpha < r$ for $r > 2$ and $t \in \mathbb{N}$.

Proof. Given the Diophantine equation $5^x + p^m n^y = z^2$. We let $y = 2$. Suppose x is an even number, such that $x = 2r$ where $r \in \mathbb{N}$, we have

$$5^{2r} + p^m n^2 = z^2. \tag{16}$$

From (16), we consider two cases depend on the possibility of the parity of m. Firstly, we let m be an even number such that $m = 2t$ where $t \in \mathbb{N}$. We have

$$5^{2r} + p^{2t} n^2 = z^2 \tag{17}$$

$$(z + p^t n)(z - p^t n) = 5^{2r-\alpha} 5^{\alpha}. \tag{18}$$

where $0 \leq \alpha \leq 2r$.
Since the LHS must be equal to RHS, we will consider all possible combinations of (18), as follows:

Table 2: Possible combinations of (18).

i	$z + p^t n = 5^r$,	$z - p^t n = 5^r$
ii	$z + p^t n = 5^{2r}$,	$z - p^t n = 1$
iii	$z + p^t n = 1$,	$z - p^t n = 5^{2r}$
iv	$z + p^t n = 5^{2r-\alpha}$,	$z - p^t n = 5^{\alpha}$
v	$z + p^t n = 5^{\alpha}$,	$z - p^t n = 5^{2r-\alpha}$

By solving (i) simultaneously, we obtain

$$n = 0.$$

This is contradicts since n must be positive integer.

By solving (ii) simultaneously, we obtain

$$z = \frac{5^{2r} + 1}{2}. \tag{19}$$

From (iii), we have

$$z + p^t n = 1 \tag{20}$$
$$z - p^t n = 5^{2r}.$$

By solving the above equation simultaneously, we obtain

$$z = \frac{1 + 5^{2r}}{2}$$

which is similar to (19). Substitute (19) into (20), we obtain

$$n = \frac{1 - 5^{2r}}{2p^t}.$$

This is contradicts since n must be positive integer.

By solving (iv) simultaneously, we obtain

$$z = \frac{5^{2r-\alpha} + 5^{\alpha}}{2}. \tag{21}$$

By solving (v) simultaneously, we obtain

$$z = \frac{5^{\alpha} + 5^{2r-\alpha}}{2}$$

which is similar to (21).

By equations (19) and (21), we obtain

$$z = \frac{5^{2r-\alpha} + 5^{\alpha}}{2} \tag{22}$$

where $\alpha \geq 0$.

Substitute (22) into (17), we obtain

$$n = \frac{5^{2r-\alpha} - 5^{\alpha}}{2p^t}.$$

Since n must be positive integer, then $0 \leq \alpha < r$ for $r > 2$ and $t \in \mathbb{N}$. □

Now, from (16), we let m be an odd number. To solve this Diophantine equation, we consider the following corollary.

Corollary 2.1. : *Let x, m, n, y, z be positive integers and $p > 5$ a prime number. If x is an even number, m is an odd number and $y = 2$, then the fundamental solution for n and z in the Diophantine equation $5^x + p^m n^y = z^2$ must satisfy the following inequalities*

$$0 < n \leq \frac{5^r b_1}{\sqrt{2(a_1 + 1)}},$$

$$0 <| z |\leq \sqrt{\frac{5^{2r}(a_1 + 1)}{2}}$$

with

$$(x, m) = (2r, 2t - 1)$$

for $r, t \in \mathbb{N}$ *where* (a_1, b_1) *is a fundamental solution of* $z^2 - Dn^2 = 1$ *and* $D = p^{2t-1}$.

Proof. Given the Diophantine equation $5^x + p^m n^y = z^2$, we let $y = 2$, suppose x is an even number and m is an odd number such that $x = 2r$ and $m = 2t - 1$ where $r, t \in \mathbb{N}$, we have

$$z^2 - p^{2t-1} n^2 = 5^{2r}.$$

Since p^{2t-1} is not a perfect square, let $p^{2t-1} = D$. We obtain

$$z^2 - Dn^2 = 5^{2r}. \tag{23}$$

Refer to Theorem 1.1, the fundamental solution for n and z in (23) must satisfy the following inequalities

$$0 < n \leq \frac{5^r b_1}{\sqrt{2(a_1 + 1)}},$$

$$0 <\mid z \mid\leq \sqrt{\frac{5^{2r}(a_1 + 1)}{2}}$$

for $r \in \mathbb{N}$ where (a_1, b_1) is a fundamental solution of $z^2 - Dn^2 = 1$. □

3. Conclusion

The integral solutions to the Diophantine equation $5^x + p^m n^y = z^2$ are as follow:

1. For $y = 1$, we obtain

$$(x, m, n, y, z) = (2r, t, p^t k^2 \pm 2k5^r, 1, p^t k \pm 5^r)$$

   where $k, r, t \in \mathbb{N}$.

2. For $y = 2$ and m is even number, we obtain

$$(x, m, n, y, z) = \left(2r, 2t, \frac{5^{2r-\alpha} - 5^\alpha}{2p^t}, 2, \frac{5^{2r-\alpha} + 5^\alpha}{2}\right)$$

where $0 \leq \alpha < r$ for $r > 2$ and $t \in \mathbb{N}$.

3. For $y = 2$ and m is odd number, we obtain

$$(x, m) = (2r, 2t - 1)$$

with

$$0 < n \leq \frac{5^r b_1}{\sqrt{2(a_1 + 1)}},$$

$$0 <| z |\leq \sqrt{\frac{5^{2r}(a_1 + 1)}{2}}$$

where $r, t \in \mathbb{N}$ and (a_1, b_1) is a fundamental solution of $z^2 - Dn^2 = 1$ where $D = p^{2t-1}$.

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